ST402 Principals and Methods of Statistical Practice
Mid-term Test: Solutions

1. I have an urn with 3 disks of which 2 are green and 1 is red. I toss a fair coin twice and count the number of heads; X is the number of heads. I then draw X disks from the urn without replacement; Y is the number of red disks that I draw.

   (a) Find $E(X)$

   X has a binomial distribution with $n = 2$ and $\pi = 1/2$, so $EX = n\pi = 1$.

   and $E(Y|X)$

   Clearly, $E[Y|X = 0] = 0$.

$$E[Y|X = 1] = 0 \times P[Y = 0|X = 1] + 1 \times P[Y = 1|X = 1]$$
$$= P[Y = 1|X = 1] = 1/3.$$

$$E[Y|X = 2] = 0 \times P[Y = 0|X = 2] + 1 \times P[Y = 1|X = 2]$$

$$= P[Y = 1|X = 2] = 2/3.$$

   So, we can summarise as $E[Y|X] = X/3$.

   Hence find $E(Y)$.

$$E[Y] = E[E[Y|X]] = E[X/3] = E[X]/3 = 1/3.$$

   (b) Find the probability mass function of Y.

$$P(Y = 0) = P(Y = 0|X = 0)P(X = 0)+P(Y = 0|X = 1)P(X = 1)+P(Y = 0|X = 2)P(X = 2)$$
$$= 1 \times 1/4 + 2/3 \times 1/2 + 1/3 \times 1/4 = 2/3.$$

$$P(Y = 1) = P(Y = 1|X = 1)P(X = 1) + P(Y = 1|X = 2)P(X = 2)$$
$$= 1/3 \times 1/2 + 2/3 \times 1/4 = 1/3.$$

2. Let X and Y be random variables with joint density function

$$f_{X,Y}(x, y) = \begin{cases} kx, & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

(a) Find k.

The marginal density function of Y for $0 < y < 1$ is

$$f_Y(y) = \int_0^y kx dx = k[x^2/2]_0^y = ky^2/2.$$

for other values of y, $f_Y(y) = 0$. So, for total probability of 1,

$$1 = \int_0^1 ky^2/2dy = [ky^3/6]_0^1 = k/6.$$

Hence $k = 6$.

(b) Write down the density of $X|Y = y$

the conditional density function, for $0 < x < y$, and $0 < y < 1$, is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{F_Y(y)} = 6x/3y^2 = 2x/y^2.$$

and for other values of x that conditional density function is zero. For other values of y it is not defined.

and hence find $\text{Var}(X|Y = \frac{1}{2})$.

For $0 < y < 1$

$$E[X|Y = y] = \int_0^y x f_{X|Y}(x|y)dx$$

$$= \int_0^y x2x/y^2 dx = \left[\frac{2x^3/3}{y^2}\right]_0^y = 2y/3.$$

$$E[X^2|Y = y] = \int_0^y x^2 f_{X|Y}(x|y)dx$$

$$= \int_0^y x^2 2x/y^2 dx = \left[\frac{2x^4/4}{y^2}\right]_0^y = y^2/2.$$

So $\text{Var}[X|Y = y] = y^2/2 - (2y/3)^2 = y^2/18$, and $\text{Var}[X|Y = 1/2] = 1/72$

3. Let Y and Z be independent standard normal random variables.

(a) Show that the moment generating function of Y is $M_Y(t) = e^{t^2/2}$

$$\begin{aligned} M_Y(t) &= \int_{-\infty}^{\infty} \exp(yt)\frac{1}{\sqrt{2\pi}}\exp(-y^2/2)dy \\ &= \int{}_{-\infty}{}^{\infty}\frac{1}{\sqrt{2\pi}}\exp(-frac12[(y-t)^2+t^2]dy \\ &= \exp(t^2/2), \end{aligned}$$

as the integral is over the whole range of a density function for $N(t, 1)$.

and hence write down the joint MGF $M_{Y,Z}(t, u)$

As Y, Z are independent and have the same distribution,

$$M_{Y,Z}(t, u) = M_Y(t)M_Z(u) = \exp(\frac{1}{2}(t^2 + u^2)).$$

(b) Let $W = Y$ and $X = \rho Y + \sqrt{(1-\rho^2)}Z$. Derive the joint moment generating function of W and X.

$$\begin{aligned} M_{W,X}(t, u) = E\exp[tW + uX] &= E\exp[tY + u\rho Y + u\sqrt{(1-\rho^2)}Z] \\ &= E\exp[Y(t + u\rho) + Zu\sqrt{(1-\rho^2)}] \\ &= \exp[\frac{1}{2}[(t + u\rho)^2 + (u\sqrt{(1-\rho^2)})^2]] \\ &= \exp[\frac{1}{2}[t^2 + 2tu\rho + u^2]]. \end{aligned}$$

Show that X is a standard normal random variable.

Putting $t = 0$ in the above gives the MGF for X as $\exp[-u^2/2]$, which identifies it as a standard normal random variable.

4. Suppose that $X_j \sim iN(0, \sigma_X^2)$ and $Y_j \sim iN(\mu_Y, \sigma_Y^2)$ for $j = 1, \ldots, n$.

(a) Find the r^{th} moment of X.

We have $M_X(t) = \exp(\sigma_X^2 t^2/2)$, so expanding in an exponential series,

$$M_X(t) = \sum_{j=0}^{\infty} \sigma_X^{2j} t^{2j}/j!/2^j$$

and the moments μ_r being the coefficients of $t^r/r!$ are given by

$$\mu_r = r!\sigma_X^r/(2^{r/2}(r/2)!$$

for r even, and $\mu_r = 0$ for r odd.

(b) Find the expected value and variance of the estimator of σ_X^2

$$\hat{\sigma}_X^2 = \frac{1}{n}\sum_{j=1}^{n} X_j^2$$

We have $\hat{\sigma}_X^2$ the mean of n independent and identically distributed random variables, so

$$E[\hat{\sigma}_X^2] = E[X_1^2] = \mu_2 = \sigma_X^2.$$

$$\mathrm{Var}\hat{\sigma}_X^2 = \mathrm{Var}X_1^2/n = (\mu_4 - \mu_2^2)/n = (3\sigma_X^4 - \sigma_X^4)/n = 2\sigma_X^4/n.$$

(c) Find the expected value of the the estimator of σ_Y^2

$$\hat{\sigma}_Y^2 = \frac{1}{n}\sum_{j=1}^{n}(Y_j - \overline{Y})^2$$

where $\overline{Y} = \frac{1}{n}\sum_{j=1}^{m} Y_j$.

We can use $Y_j - \bar{Y} = W_j - \bar{W}$ where $W_j = Y_j - \mu_Y\ iN(0, \sigma_Y^2)$. Then, $\hat{\sigma}_Y^2 = \frac{1}{n}\sum_{j=1}^{n} W_j^2 - n\bar{W}^2$, and

$$E[\hat{\sigma}_Y^2] = \sigma_Y^2 - \sigma_Y^2/n = \frac{n-1}{n}\sigma_Y^2.$$